Journal of Fluids and Structures 26 (2010) 467–481

JOURNAL OF FLUIDS AND STRUCTURES

www.elsevier.com/locate/jfs

A mathematical simulation to synchronization of a bluff structure

R.G. Williams[a,*], W. Suaris[b], R.H. Goodman[c]

[a]Department of Civil and Environmental Engineering, University of the West Indies, Saint Augustine, Trinidad and Tobago, West Indies

[b]Department of Civil, Architectural and Environmental Engineering, University of Miami, Coral Gables, Miami, FL, USA

[c]Department of Mathematics, University of Miami, Coral Gables, Miami, FL, USA

Received 29 October 2008; accepted 15 December 2009
Available online 12 February 2010

Abstract

When the periodic vortex-shedding frequency of a bluff structure "locks-in" with one of the natural frequencies of the structure, the state is termed synchronization. When this lock-in or detuning state persists over a considerable period of time, fatigue stresses become predominant which then may lead to structural damage and even failure.

Vortex-excited oscillations of bluff structures are one of the important problems in wind engineering, and wake oscillator models, especially the Van der Pol or Rayleigh type, have been studied profoundly over the last 20 odd years. The approach of most researchers is to couple one form of the Rayleigh oscillator with the conventional equation of motion of a single degree of freedom system. It is emphasized though that, depending on the form of the Rayleigh equation chosen, convergence to the solution may not be guaranteed, and the problem becomes ill-defined. The proposed semi-empirical model is of the coupled Rayleigh wake-oscillator type with the equation of motion containing a series term for the forcing function. The paper highlights the mathematical suitability of the proposed form for the aerodynamic response of an isolated structure.

Keywords: Vortex; Frequency; Aerodynamic response; Rayleigh; Van der Pol

1. Introduction

The separation of a uniform two-dimensional flow due to its encounter with a bluff body in its path results in the formation of the two zones of concentrated vorticity, across which a large velocity gradient exists. These are referred to as the shear layers. Fed by the energy in the oncoming flow, the vorticity in the shear layers keeps growing until it reaches a critical stage when the two shear layers interact and shed the vorticity in the form of a vortex that convects downstream. This sequence is repeated periodically with the formation of the familiar vortex street (Simiu and Scanlan, 1996).

For a rigidly fixed cylinder, an alternating, staggered vortex street adequately portrays the phenomenon of vortex shedding downstream of the cylinder. But the consideration of an elastic system that can move is a more difficult task.

*Corresponding author. Tel.: +1 868 662 2002x3433.
E-mail address: rupert.williams@sta.uwi.edu (R.G. Williams).

0889-9746/$ - see front matter
doi:10.1016/j.jfluidstructs.2009.12.003

Nomenclature

a	$\rho D^2 l/(8\pi^2 S^2 M)$
C	total linear damping constant
c_L	instantaneous lift coefficient in τ domain
D	diameter of cylinder
F_a	external lift force
F_L	lift force
$f(x)$	forcing function
$f(c_L)$	forcing lift force
f_s	frequency of vortex shedding, SV/D
f_n	fundamental frequency of body
K_e	total stiffness of springs
k_c	stiffness of cylinder in lift direction
K_L	constant value relating to lift force
L	characteristic length of cylinder
$\overline{L}$	Lipschitz constant
l	length of cylinder
M	mass of cylinder
S	Strouhal number
Sr	Scruton number, $m\zeta/\rho D^2$
V	fluid stream velocity
v	wind velocity
V_{air}	volume of air
x	lift displacement
x_r	dimensionless lift displacement variable, x/D
α	damping constant in Rayleigh equation
γ	damping constant
δ	phase shift
$\overline{\varepsilon}$	weighted constant for forcing function based on the value $(4M/L)/(\pi\rho_{\mathrm{air}}DL)$
ε	constant denoting small change
ε_0	proportionality constant
ε_1	constant related to mass of cylinder
ε_2	constant related to mass of fluid medium
ζ	damping ratio of cylinder, $C/2M\omega_n$
ρ	density of fluid medium
ρ_{air}	density of air
τ	$\omega_n t$
ω_0	f_s/f_n
ω_n	natural circular frequency of body, $2\pi f_n$
ω_s	Strouhal circular frequency, $2\pi f_s$
(')	first order derivative with respect to τ, $d(\ )/d\tau$
$(\ )'$	material time derivative, $d(\ )/dt$
$(\ )''$	second order derivative, $d^2(\ )/dt^2$

In addition to the flow interactions leading to the vortex street, there occurs a highly complex feedback effect from the wake to the elastically supported cylinder. The degree of influence of this feedback upon the phenomenon depends on the proximity of the Strouhal frequency, f_s, to the natural frequency of the elastic system, f_n. Over a certain range of windspeeds, for which the detuning or separation of these two frequencies is close to zero, the periodicity in the wake locks-in or is synchronized by that of the mechanical system. The shedding frequency abruptly deviates from varying linearly with the stream velocity, as depicted in Fig. 1, and stays constant with that of the mechanical system. In this state, the feedback from the synchronized wake to the cylinder intensifies and the response amplitude of the elastically supported cylinder grows to a maximum limit marking the end of the synchronization range. This phenomenon was studied by Bearman (1984), who is just one of many researchers in the area of vortex shedding.

The Strouhal frequency is defined as

$$f_s = \frac{SV}{D}, \tag{1}$$

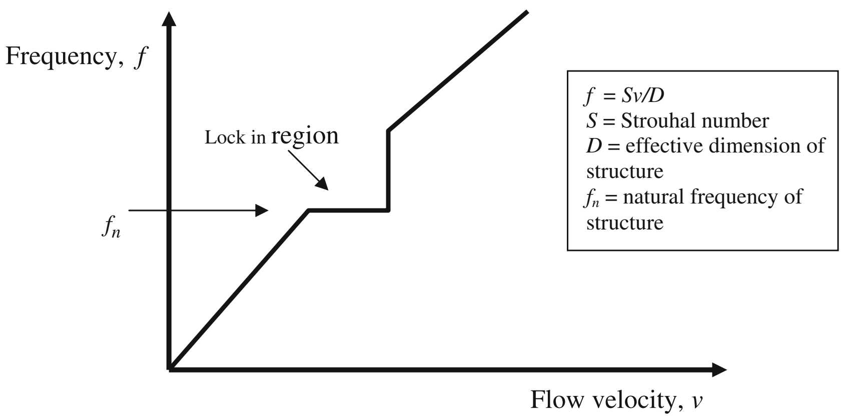

Fig. 1. Strouhal frequency, fs versus flow velocity, v.

where V is the incident wind (fluid) speed, D the lateral dimension of the body, and S the Strouhal number. The Strouhal number depends on the geometric properties of the body (Ohya et al., 1989).

In experimental work, vortex-induced vibration may be grouped into two categories, depending on the fluid in use, i.e. air or water. Due to a lower mass ratio (fluid to structure) in the case of air, response amplitudes are considerably lower than for analogous experiments with water. Further classification is required, depending on whether the cylinder is allowed to vibrate freely during the loading states or whether it is subjected to forced vibrations where the frequency and amplitude of vibration are controlled independently. Independent control over the frequency and amplitude of cylinder oscillation makes it easier to observe the effect of variation of either parameter. For free-vibration experiments, the vibration amplitude and frequency of vortex shedding keep changing simultaneously as the stream velocity is varied. In the forced vibration experiments, however, the cylinder vibration may no longer be called vortex-induced, since the phenomenon is altered from its free state by steady state impulses. It is interesting to note here that the reports of several investigators indicate that for a certain pair of values of stream velocity, V, and damping ratio, ζ, within the synchronization range, the vortex-induced response is bi-stable, that is, the system may respond at either of two distinct, stable amplitudes. The actual response amplitude is determined by the initial conditions (Shih et al., 1993).

2. Previous research

2.1. Analytical

Wake oscillator models based on the Rayleigh equation or more commonly the Van der Pol oscillator equation can be placed into two broad categories, based generally on the form of the Rayleigh equation employed. Within these two categories, there can be further subdivision based on the forcing function of the oscillator, i.e. displacement coupled, velocity coupled, or acceleration coupled.

Now, starting with the more common system, as employed by Skop and Balasubramanian (1997), this model employs a velocity-coupled system based on the classical Van der Pol equation. Their model, as given by Eqs. (17) and (18) in Skop and Balasubramanian, is as follows:

$$\ddot{y} + 2\zeta_i\omega_{n,i}\dot{y} + \omega_{n,i}^2 y = \mu\omega_s^2\left(q - \frac{2\alpha}{\omega_s}\dot{y}\right), \tag{2}$$

$$\ddot{q} - \omega_s G(C_{L0}^2 - 4q^2)\dot{q} + \omega_s^2 q = \omega_s F\dot{y}. \tag{3}$$

The equations above are reproduced using the same symbols and definitions as in Skop and Balasubramanian (1997).

Now considering the velocity-coupled system above, by making reference to the lift phenomenon, physically it may be explained, but mathematically the formulation is weak, since it is incomplete, and may lead to divergence as a result of secular terms in the series expansion. A more traditional and accepted approach would have been to use a truncated series expression incorporating the velocity term. This secularity could be one reason why the stall term had to be added to the forcing function of the equation of motion, Eq. (2), to counter the low damping divergence produced.

The second model is that of Facchinetti et al. (2004). This model employs an acceleration-coupled system and is given in Eq. (6) of Facchinetti et al., as follows:

$$\ddot{y} + \left(2\xi\delta + \frac{\gamma}{\mu}\right)\dot{y} + \delta^2 y = s, \tag{4}$$

$$\ddot{q} + \varepsilon(q^2 - 1)\dot{q} + q = f, \tag{5}$$

Again these equations are reproduced using the same symbols and definitions as in Facchinetti et al. (2004).

It is to be noted here that this model is based on acceleration-coupling using a single term as defined by f in Eq. (5). Now, even if this is explained based on energy principles, for the reasons stated above, this may lead to instability and divergence. It would have proven to be a more complete and compatible formulation if again a truncated series was used incorporating the second-order acceleration term. This would have also ensured continuity.

Now, considering Eqs. (3) and (5), we see the form of the Van der Pol equation used has the nonlinear term, $\dot{q}\, q^2$. And because of this particular nonlinear term, it can be shown that this form of the oscillator equation doe not satisfy the Lipschitz condition; from the theory put forward in the succeeding sections, the Lipschitz condition is one condition to guarantee that the problem be well-defined and produce a solution. Moreover when dealing with transcendental

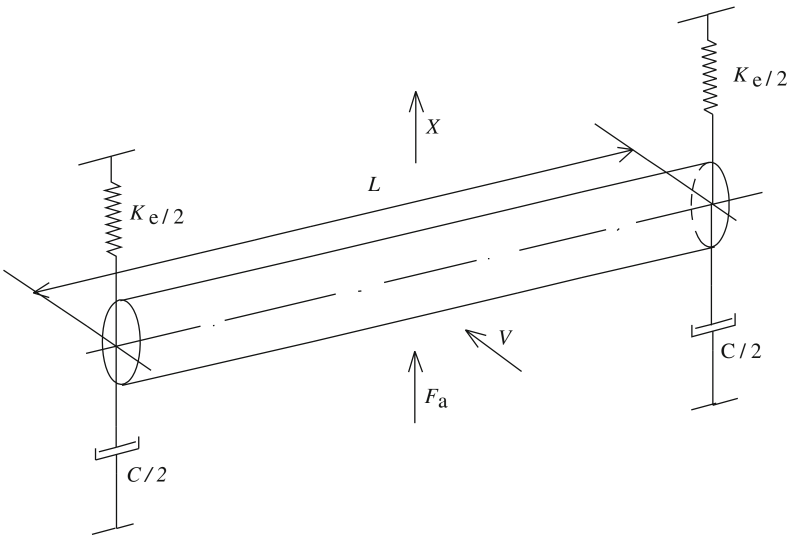

Fig. 2. Model structure for analysis.

equations, such as the Rayleigh equations, we need to ensure that the problem is well-defined in order that a numerical solution exists. For ill-defined problems, in progressing towards the solution, as the time steps are reduced, the results produced may not be meaningful, whether we have velocity coupling or acceleration-coupling, as instability will occur.

The third model is the model proposed in the current paper and is elaborated in Williams and Suaris (2006). The model stems from that of Hartlen and Currie (1970), and forms the foundation of the present work. The physical system considered is shown in Fig. 2. The cylinder of diameter D and length l is mounted elastically on springs of total stiffness K_e, and the total linear damping constant is C, and M is the cylinder mass. The external lift force acting on the cylinder is F_a, with x being the amplitude in the lift direction. We now introduce dimensionless variables x_r and τ defined as

$$x_r = (x/D), \quad \tau = t\left(\frac{K_e}{M}\right)^{1/2} = \omega_n t \tag{6, 7}$$

and dimensionless parameters defined by

$$\zeta = \frac{C}{2M\omega_n}, \quad a = \frac{\rho D^2 l}{8\pi^2 S^2 M}, \quad \omega_0 = \frac{f_s}{f_n} = S\left(\frac{V}{f_n D}\right), \tag{8, 9, 10}$$

where S is the Strouhal number for the cylinder, V the velocity of the fluid flow with ρ being the density of the fluid; f_s the vortex shedding frequency which is related to the Strouhal number, S, and f_n the natural frequency of the cylinder on its elastic mounting. The characteristic equation of dynamic equilibrium of the cylinder in dimensionless form becomes

$$\ddot{x}_r + 2\zeta\dot{x}_r + x_r = a\omega_0^2 c_L, \tag{11}$$

where c_L is the instantaneous lift coefficient that is influenced by the motion of the cylinder; therefore it is a function of τ. The overdot indicates differentiation with respect to τ.

The second part of the mathematical model involves the development of an equation for c_L. The approach taken is well laid out in Hartlen and Currie (1970). The form of the equation for c_L will be

$$\ddot{c}_L - \alpha\omega_0\dot{c}_L + \frac{\gamma}{\omega_0}(\dot{c}_L)^3 + \omega_0^2 c_L = \text{forcing term} \tag{12}$$

with α and γ being damping terms.

Eqs. (11) and (12) represent the coupled wake oscillator model as developed by Hartlen and Currie.

3. Proposed model

3.1. Wake oscillator model for isolated cylinder

This section is discussed in detail in Williams and Suaris (2006). Writing the equation of motion in its basic form, we have the following:

$$x'' + \text{damping term} + \omega_n^2\, x = \varepsilon_I f(c_L), \tag{13}$$

where x is the displacement in the lift direction, $f(c_L)$ is a forcing function related to the lift (wind) coefficient, c_L, and ε_1 is a constant related to the mass of the cylinder, with ω_n being the natural frequency of the cylinder. The prime denotes differentiation with respect to time, t.

Now considering the lift force function, $f(c_L)$ as a product of the static pressure and the cross-sectional area of the cylinder, i.e. $\frac{1}{2}\rho v^2 DLc_L$ and $\varepsilon_1 = M^{-1}$, the equation is reduced to

$$x'' + \text{damping term} + x = (1/M)DL\tfrac{1}{2}\rho v^2 c_L, \tag{14}$$

where M is the mass of the cylinder of diameter D, and length L, ρ the density of the fluid medium, v the velocity of the fluid stream, and c_L the lift (wind) coefficient, with ω_n being the natural frequency of the cylinder.

Expressing the aerodynamic equation in a similar form, we have

$$c_L'' + \text{damping term} + \omega_s^2\, c_L = \varepsilon_2 \frac{f(x)}{L}, \tag{15}$$

where $f(x)/L$ is a forcing function per unit length related to the motion of the cylinder. The forcing function is expressed per unit length to be consistent with the final expression for the forcing function in the equation of motion and also for dimensional consistency of the equation. By analogy with Eq. (13), ε_2 is a function of the wind mass, represented by ρ_{air} V_{air}, the product of the density of the fluid medium and the contributing volume of air; $\omega_s = 2\pi f_s$ is the Strouhal circular frequency.

From analyses carried out, and using a series expression for the forcing function, the lift coefficient, c_L, was observed to be proportional to the amplitude of vibration, before and up to the end of the synchronization region. This behavior is consistent with the results of the experiments of Bishop and Hassan (1964).

Introducing $\tau = \omega_n t$ and noting that

$$\frac{d^2 c_L}{dt^2} = \omega_n^2 \frac{d^2 c_L}{d\tau^2}, \quad \frac{d^2 x_r}{dt^2} = \omega_n^2 \frac{d^2 x_r}{d\tau^2}.$$

Eqs. (14) and (15) can be written using dimensionless parameters in the τ domain as

$$\ddot{x}_r + \text{damping term} + x_r = (1/M)L\frac{\frac{1}{2}\rho v^2 c_L}{\omega_n^2}, \tag{16}$$

$$\ddot{c}_L + \text{damping term} + \omega_0^2 c_L = \varepsilon_2 \frac{f(x_r)D}{\omega_n^2 L}, \tag{17}$$

where x_r is the dimensionless lift displacement x/D, and $\omega_0^2 = \omega_s^2/\omega_n^2$.

Setting up a recurrence relation for $f(x_r)$, we have

$$f(x_r) = k_c(x_r + \Delta\tau.\dot{x}_r), \tag{18}$$

where k_c is the stiffness of the cylinder in the lift direction. This form is chosen since x_r is the lift displacement; therefore, the lift force is a product of this ‘‘displacement’’ and the corresponding stiffness. A truncated series expression is used in lieu of a single term to avoid secular definitions which may lead to divergence.

Now recalling that $k_c/\omega_n{}^2$ is the mass (M) of the cylinder and assuming the contributing volume of air is proportional to the volume of the cylinder, Eq. (17) can be written in a dimensionally consistent form as follows:

$$\ddot{c}_L + \text{damping term} + \omega_0^2 c_L = \varepsilon_0 \frac{4M}{\pi\rho_{\text{air}} DL^2}(x_r + \Delta\tau\ddot{x}_r), \tag{19}$$

where the term ε_0 is a proportionality constant, since the contributory volume of air was assumed to be proportional to the cylinder volume. It should be noted that the constant term on the right-hand side of Eq. (19) bears a similarity to the Scruton number, except for the damping ratio.

The two equations given below are chosen to represent the isolated system. They are the general second-order equation of motion and the modified Rayleigh equation:

$$\ddot{x}_r + 2\zeta\dot{x}_r + x_r = a\omega_0^2 c_L, \tag{20}$$

$$\ddot{c}_L - \alpha\omega_0\dot{c}_L + \frac{\gamma}{\omega_0}\dot{c}_L^3 + \omega_0^2 c_L = \varepsilon_0 \frac{4M}{\pi\rho_{\rm air}DL^2}(x_r + \Delta\tau\dot{x}_r). \tag{21}$$

These equations are similar to the system of equations developed by Hartlen and Currie (1970), with the exception of the forcing term in the aerodynamic equation being modified. The parameters α and γ are damping terms, with α being a negative damping term and γ a positive damping term. The forcing function of the aerodynamic Eq. (20) is related to the motion of the cylinder. The constant $a = \rho D^2 l/(8\pi^2 S^2 M)$, where S is the Strouhal number.

The damping function in the aerodynamic equation (21) is used in preference to the classical Rayleigh equation since the limit cycle in this form is independent of the reduced velocity, ω_0, whereas the classical model shows some dependence on the reduced velocity. The classical form also has one damping constant, whereas the form chosen here has two damping terms, which gives greater flexibility in the evaluation of the parameters to match the various experimental data available.

The model developed here is that proposed by Williams and Suaris (2006). The method used for the solution of the system of equations was the finite difference approximation. A parametric study was carried out on the Rayleigh equation using the Euler method. From this analysis, the limit cycle was found to be independent of the initial conditions, c_{L0}, and dependent solely on α and γ. In applying the finite difference approximation to the solution of the system equations (20) and (21), the values for x_{-1} and $\ddot{x}_0$ were generated from recurrence relationships. A Fortran 90 compiler was utilized to generate the algorithms. The time step used, along with the precision of the compiler eliminated the presence of nonlinear instability due to manipulation of small numbers.

3.2. Parametric study on the modified Rayleigh oscillator

The form of the Rayleigh equation chosen is as follows:

$$\ddot{c}_L - \alpha\omega_0\dot{c}_L + \frac{\gamma}{\omega_0}\dot{c}_L^3 + \omega_0^2 c_L = 0. \tag{22}$$

The numerical solution represented in the phase plane for various initial conditions is shown in Figs. 3–6. In the figures, c_L represents the value of the lift coefficient, and $\dot{c}_L$ its first derivative with respect to τ. The values of the parameters, α γ and ω_0 are set at unity. The results of this part of the analysis indicate that the radius of the limit cycle is approximately 1.25 for the parameter values chosen, and it is independent of the initial conditions of the system. This observation contrasts to the Hartlen and Currie (1970) relationship, where the solution was dependent on the initial conditions of the system. A similar analysis, keeping the initial condition constant along with the parameters α and γ, and varying the frequency of the system ω_0, indicates that the limit cycle is again independent of ω_0. This highlights the robustness of the chosen form of the Rayleigh equation and confirms its suitability for use in the development of an aerodynamic lift response model.

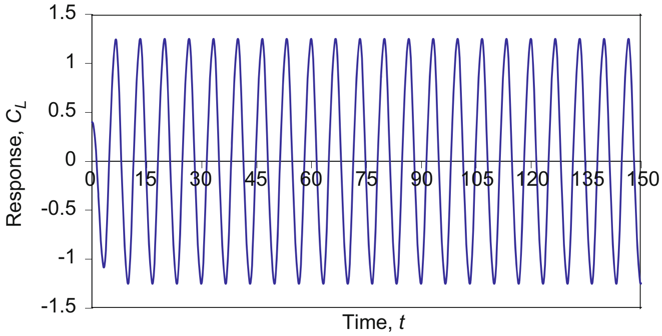

Fig. 3. System response with initial condition, $C_{L0} = 0.4$; lift response, C_L, versus time, τ.

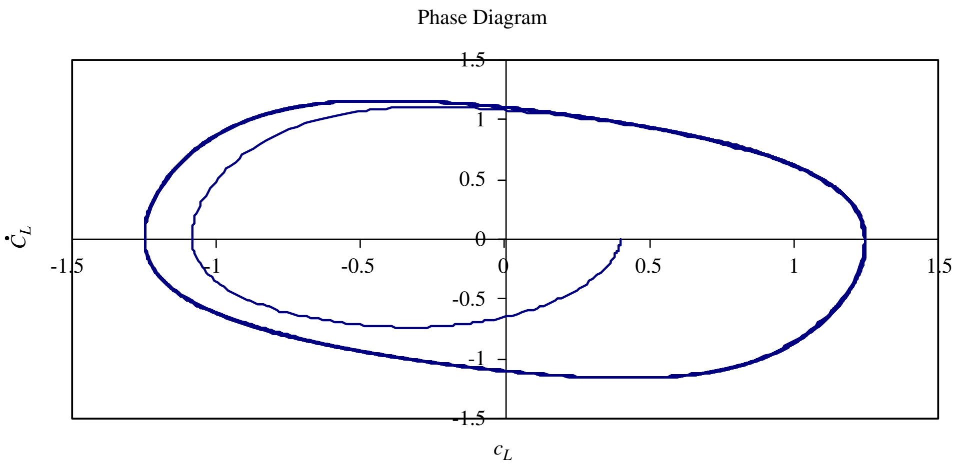

Fig. 4. Phase plane plot of system response with initial condition, $C_{L0} = 0.4$.

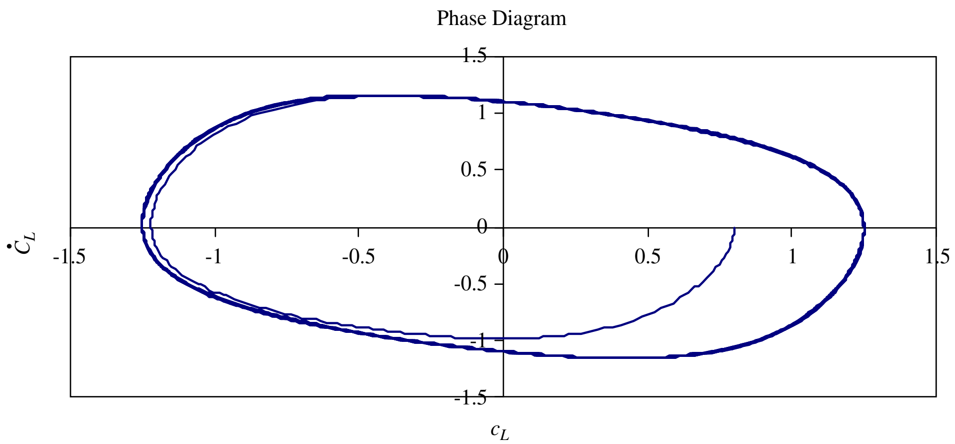

Fig. 5. Phase plane plot of system response with initial condition, $C_{L0} = 0.8$.

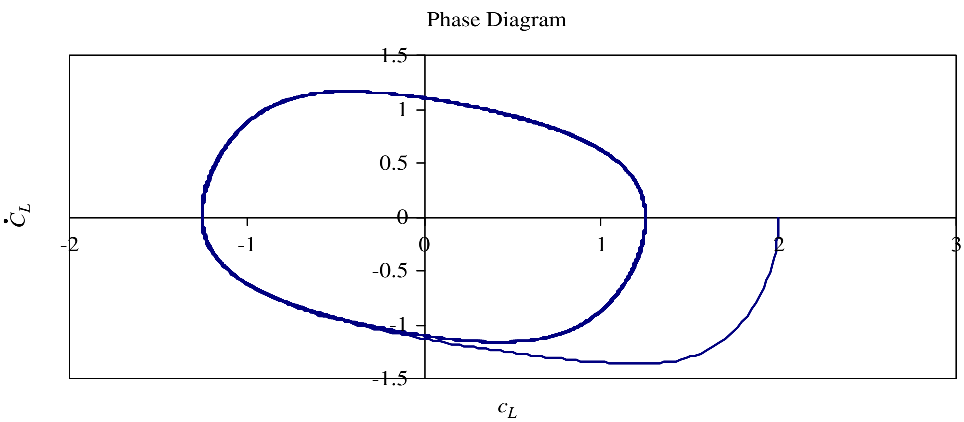

Fig. 6. Phase plane plot of system response with initial condition, $C_{L0} = 2.0$.

The relaxation oscillations of this nonlinear system are persistent; therefore models incorporating its use will tend to be structurally stable. From the analysis above, the limit cycle was a permanent feature of the phase portrait, even with the introduction of small perturbations. This insensitivity to small changes indicates that the model equation is robust.

The bifurcation of the equation was, however, evident as the parameters α and γ were varied while keeping the initial conditions and the frequency ω_0 constant. The limit cycle grew in size away from the fixed point as the parameter α increased. This Hopf bifurcation is consistent with the behavior of the Rayleigh equation. The γ term behaved as a positive damping term. Therefore, in using the Rayleigh oscillator for the proposed model, only variations in the parameters α and γ are considered when matching the results of various experimental series.

3.3. Calibration of the model parameters

The experimental work of Brika and Laneville (1993), Kitagawa et al. (1997), and Goswami et al. (1993), was used to evaluate the parameters α and γ. Expressing Eq. (21) in a form that is numerically manageable, the constant term ε_0 is chosen so as to make the coefficient unity for Goswami's experimental series. The values for all the other tests were pro-rated in accordance with their respective values of the term, $(4M/L)/(\pi\rho_{\mathrm{air}}DL)$.

The terms α and γ are damping terms, with α, being a negative damping term that affects the location of the maximum amplitude and γ a positive damping term. Both these terms were taken as being related to the Scruton number, Sr, which is related to the damping ratio. The Scruton number was chosen in preference to the damping ratio since its value provides greater flexibility, as its size is numerically manageable.

The Scruton number is defined as

$$\mathrm{Sr} = m\zeta/(\rho D^2), \tag{23}$$

where m is the mass per unit length of the cylinder, D the diameter of the cylinder, ζ the damping ratio, and ρ the density of the fluid medium. This number reflects the effect of the fluid–structure mass ratio and the level of mechanical damping in the system.

Fig. 7 shows the variation in maximum amplitude, $X_{\max}$, with the Scruton number. The experimental values are taken from previous investigations.

Once the values of the parameters have been identified for the relevant experiments, a simple regression analysis was carried out using the Scruton number as the independent variable. From this analysis, the parameters α and γ were established as

$$a = 2.13\mathrm{e}^{-\mathrm{Sr}}, \tag{24}$$

$$\gamma = 0.0136\,\mathrm{e}^{\mathrm{Sr}} + 0.36. \tag{25}$$

Figs. 8 and 9 show the results of the analytical model compared with the experimental results of Goswami et al. (1993), for two damping ratios.

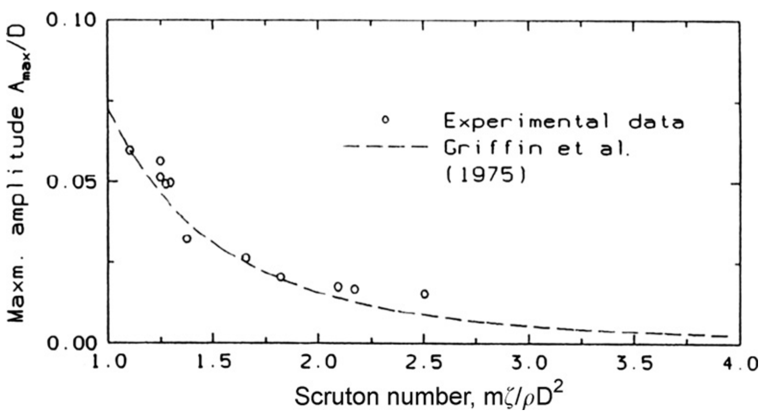

Fig. 7. Maximum amplitude, $X_{\max}$ versus the Scruton number. *Source*: Williams and Suaris (2006).

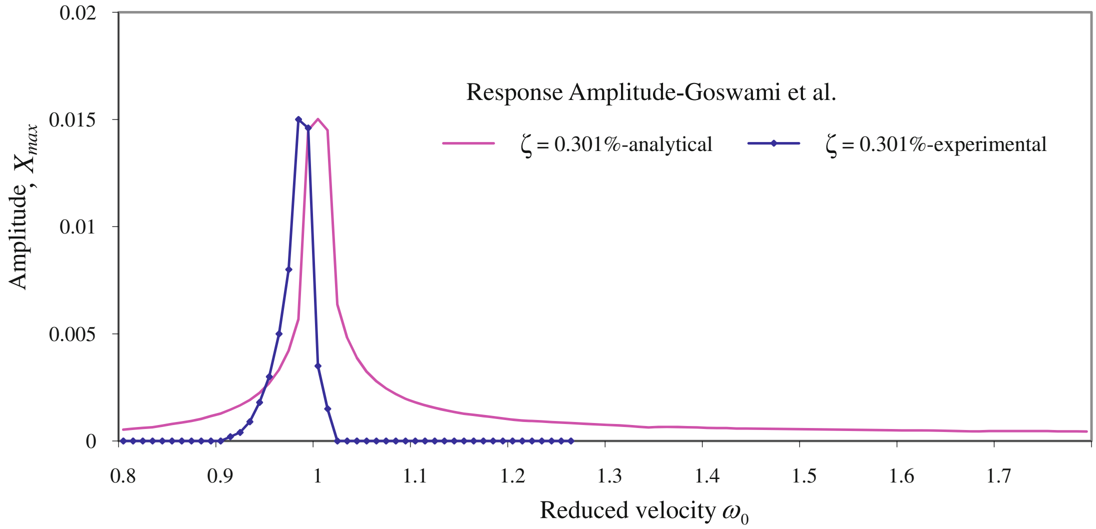

Fig. 8. Response amplitude X_{max} versus ω_0 for analytical and empirical model. Experimental data taken from Goswami et al. (1993). —, $\zeta = 0.301\%$ – analytical; —◆—, $\zeta = 0.301\%$ – experimental.

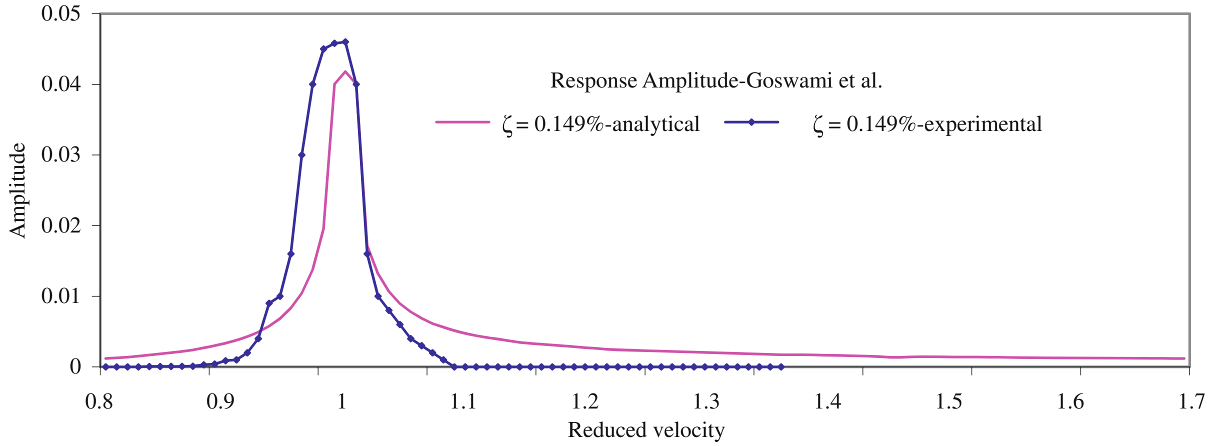

Fig. 9. Response amplitude X_{max} versus ω_0 for analytical and empirical models. Experimental data taken from Goswami et al. (1993). —, $\zeta = 0.149\%$ – analytical; —◆—, $\zeta = 0.149\%$ – experimental.

4. Mathematical existence of solution

This section covers the properties of the model equation that make it suitable for simulating the aerodynamic behavior of mechanical systems. It is to be noted that the methods that were used for solving the ordinary differential equations include Euler's method and the finite difference approximation. This section considers some of the important mathematical concepts for the existence and the uniqueness of solutions. As a matter of interest, with modern day approaches, such as the finite element technique, most partial differential equations can be reduced to a set of ordinary differential equations, so that the methods used in this paper can be applied to the solution of most analytical problems.

4.1. Continuity

Every differentiable function is continuous, but the converse is not always true. One example of this is the function, $f(x) = |x|$. This function is continuous, but it is not differentiable. So, for our present problem of examining aerodynamic behavior of a structure, represented by

$$\ddot{x}_{r-1} + 2\zeta_1\dot{x}_{r-1} + x_{r-1} = a_1\omega_{0-1}^2 c_{L-1}, \tag{26}$$

$$\ddot{c}_{L-1} - \alpha\omega_{0-1}\dot{c}_{L-1} + \frac{\gamma_1}{\omega_{0-1}}(\dot{c}_{L-1})^3 + \omega_{0-1}^2 c_{L-1} = \varepsilon f(x_{r-1}), \tag{27}$$

it has been established that the variables in these equations are differentiable, and therefore continuous. In order to prove the above statement, let us consider a function $f(\tau)$, which is differentiable at $\tau = a$. This implies that $\exists \dot{f}(a)$, such that

$$\lim_{\tau\to a}\frac{f(\tau)-f(a)}{\tau-a} = \dot{f}(a). \tag{28}$$

Then for any $\varepsilon > 0$, $\exists$ a value $\delta > 0$ such that

$$\dot{f}(a) - \varepsilon < \left|\frac{f(\tau)-f(a)}{\tau-a}\right| < \dot{f}(a) + \varepsilon, \tag{29}$$

whenever

$$0 < |\tau - a| < \delta. \tag{30}$$

Simplifying Eq. (29), we have

$$(\dot{f}(a)-\varepsilon)|\tau-a| < |f(\tau)-f(a)| < (\dot{f}(a)+\varepsilon)|\tau-a|. \tag{31}$$

On the right-hand side of Eq. (31) we have

$$|f(\tau)-f(a)| < \dot{\varepsilon}, \tag{32}$$

where

$$\dot{\varepsilon} = (\dot{f}(a)+\varepsilon)|\tau-a|. \tag{33}$$

This implies that $\dot{\varepsilon}$ is defined. And since $\dot{\varepsilon}$ is defined, $f(a)$ must be defined since $|f(\tau)-f(a)|$ exist. Then, it follows that $f(a)$ is defined. Therefore, we can assume that $f(\tau)$ is continuous at $\tau = a$. And it can generally be stated that for a single-valued function of τ to be continuous at $\tau = a$, three conditions must be satisfied:

(i) Condition 1: $\lim_{\tau\to a} f(\tau)$ exists.
(ii) Condition 2: The function is defined for the value $\tau = a$.
(iii) Condition 3: $\lim_{\tau\to a} f(\tau) = f(a)$.

If these conditions are not all satisfied, the function is discontinuous.

4.2. Existence and uniqueness of solutions

This section considers a system of n first order ordinary differential equations in normal form, that is

$$\begin{cases} \dot{c}_1 = C_1(c_1, \ldots, c_n; \tau), \\ \quad\vdots \\ \dot{c}_n = C_n(c_1, \ldots, c_n; \tau). \end{cases} \tag{34}$$

C_i are given functions of the $n+1$ real variables, $c_1, \ldots, c_n$, τ. Therefore, the object is to find solutions of Eq. (34), that is, sets of n functions $c_1(\tau), \ldots, c_n(\tau)$ of class ℓ^1, which satisfy Eq. (34). The functions C_i are assumed to be continuous and real-valued in a given region R of the $(n+1)$-dimensional space of the independent variables $c_1, \ldots, c_n, \tau$.

First-order systems of differential equations (Eq. (34)), provide a standard form to which all ordinary differential equations and systems of differential equations can be reduced. This can be achieved by matrix reduction as discussed in

Arrowsmith and Place (1982). Therefore, the theory outlined here for the existence and uniqueness of solutions of differential equations and systems of differential equations will be generalized for first-order systems.

Now, introducing vector notation, Eq. (34) can be reduced to a more concise form, which can be termed a normal first-order vector differential equation. This is written as

$$\frac{d\vec{c}}{d\tau} = \vec{C}(\vec{c}, \tau). \tag{35}$$

Therefore, a solution of Eq. (35) is a vector-valued function $\vec{c}(\tau)$ of a real (scalar) variable τ, such that $\dot{\vec{c}}(\tau) = \vec{C}(\vec{c}(\tau), \tau)$.

In order to make use of vector notation for systems of differential equations, we need to list a few facts about vectors in n-dimensional Euclidean space. They are as follows:

(i) addition of two vectors and multiplication of vectors by scalars are defined component-wise as in the plane and in space;
(ii) the length of a vector $\vec{c} = (c_1, c_2, ..., c_n)$ is defined as $|\vec{c}| = (c_1^2 + c_2^2 + \cdots + c_n^2)^{1/2}$;
(iii) length satisfies the triangle inequality, that is $|\vec{c} + \vec{x}| \le |\vec{c}| + |\vec{x}|$;
(iv) the dot product or inner product of two vectors is defined as $\vec{c} \cdot \vec{x} = c_1 x_1 + \cdots + c_n x_n$;
(v) the dot product satisfies the Schwarz inequality $|\vec{c} \cdot \vec{x}| \le |\vec{c}| \cdot |\vec{x}|$; and
(vi) integration, differentiation and other such functions are carried out component by component as in vector addition.

4.3. Initial value definition

The cylinder is considered initially at rest with no initial displacement, therefore

$$x_{r-1}(0) = 0, \quad \dot{x}_{r-1}(0) = 0. \tag{36}$$

The other initial conditions needed for complete definition of the problem are the time derivative of the lift coefficient for the cylinder, $\dot{c}_{L-1}(0)$. To define this value, we first consider the lift force on this upstream cylinder, $F_L(\tau)$. By definition

$$F_L(\tau) = K_L c_L(\tau), \tag{37}$$

where K_L is a constant value.

Now, using a forward difference approximation for the first time derivative of the lift coefficient, we have

$$\dot{c}_L(0) \approx \frac{c_L(\Delta\tau) - c_L(0)}{\Delta\tau}, \tag{38}$$

where $c_L(\Delta\tau)$ is the lift coefficient at time $\Delta\tau$. Intuitively, the lift force is generated only after vortices are shed by the structure. This indicates that there is a time lag between the fluid flow past the structure and the lift force applied to the structure. Therefore, for some finite time interval, equal to or less than this lag, $\Delta\tau$, $c_L(\Delta\tau)$ is equal to zero. So, for a time step, $\Delta\tau$, chosen small enough, and not equal to zero, $c_L(\Delta\tau)$ would be zero, leaving the numerator of Eq. (38) equal to zero.

Therefore, $\dot{c}_{L-1}(0)$ can be assumed zero for the cylinder, and the four initial conditions for definition of Eqs. (26) and (27) for the proposed model have been established.

4.4. Lipschitz condition

If a solution to a differential equation is found which satisfies all the boundary (initial) conditions, then it is the only solution to that equation. This is called the uniqueness theorem. In physical problems, a reasonable approach to finding solutions to differential equations would be to use a trial solution and try to force it to fit the boundary conditions. This section introduces a fundamental definition that is used in the uniqueness theorem. The definition is stated: a vector-valued function $\vec{C}(\vec{c}, \tau)$ satisfies a Lipschitz condition in a region R of (c,τ) space if and only if, for some finite Lipschitz constant $\overline{L}$,

$$|\vec{C}(c,\tau)-\vec{C}(d,\tau)| \le \overline{L}|c-d| \quad \text{if } (c,\tau)\in R,\ (d,\tau)\in R. \tag{39}$$

Both terms on the left-hand side of Eq. (39) involve the same value of τ.

For differential equations to be useful in predicting the future behavior of a physical system from its present state, their solutions must exist, be unique, and depend continuously on their initial values. When all these conditions are satisfied, the initial value problem is said to be well set. And it can be further stated that, if $\vec{C}$ satisfies a Lipschitz condition, then the vector differential equation, Eq. (35), defines a well-set initial value problem i.e. the solutions exist and are unique.

Now, here is a system, Eqs. (26) and (27) of two, second-order differential equations:

$$\ddot{x}_{r-1} + 2\zeta_1\dot{x}_{r-1} + x_{r-1} = a_1\omega_{0-1}^2 c_{L-1}, \tag{40}$$

$$\ddot{c}_{L-1} - \alpha\omega_{0-1}\dot{c}_{L-1} + \frac{\gamma_1}{\omega_{0-1}}(\dot{c}_{L-1})^3 + \omega_{0-1}^2 c_{L-1} = \varepsilon f(x_{r-1}). \tag{41}$$

Making the substitution $x_1 = x_{r-1}$, $x_2 = \dot{x}_{r-1}$, $c_1 = c_{L-1}$, and $c_2 = \dot{c}_{L-1}$, we can convert the system above to an equivalent system of four first order differential equations as follows:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = a_1\omega_{0-1}^2 c_1 - 2\zeta_1 x_2 - x_1, \tag{42, 43}$$

$$\dot{c}_1 = c_2, \quad \dot{c}_2 = \varepsilon.f(x_1) + \alpha\omega_{0-1}c_2 - \frac{\gamma_1}{\omega_{0-1}}(c_2^3) - \omega_{0-1}^2 c_1. \tag{44, 45}$$

Simplifying the above into a more general form, we have

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\tilde{D}x_2 - x_1 + \tilde{G}c_1, \tag{46, 47}$$

$$\dot{c}_1 = c_2, \quad \dot{c}_2 = Dc_2 - Ec_2^3 - Fc_1 + Gf(x_1), \tag{48, 49}$$

where $\tilde{D}$, $\tilde{G}$, D, E, F, and G represent the coefficients of the corresponding variables in Eqs. (42)–(45), and $f(x_1)$ is a general function of x_1 which can be taken as equal to x_1 in this case without loss of generality. The dot in the equations denotes the derivative with respect to τ. Eqs. (46)–(49) can be further simplified to give

$$\dot{x}_1 = X_1, \quad \dot{x}_2 = X_2, \quad \dot{c}_1 = C_1, \quad \dot{c}_2 = C_2, \tag{50, 51, 52, 53}$$

where these equations are for the cylinder with X_1, X_2, C_1, and C_2 defined as follows:

$$X_1 = x_2, \quad X_2 = -\tilde{D}x_2 - x_1 + \tilde{G}c_1, \quad C_1 = c_2, \quad C_2 = Dc_2 - Ec_2^3 - Fc_1 + Gf(x_1). \tag{54, 55, 56, 57}$$

Eqs. (54)–(57) can be reduced to

$$\vec{X}(\vec{x},\tau) = \begin{Bmatrix} X_1(x_1,x_2,\tau) \\ X_2(x_1,x_2,\tau) \end{Bmatrix}, \quad \vec{C}(\vec{c},\tau) = \begin{Bmatrix} C_1(c_1,c_2,\tau) \\ C_2(c_1,c_2,\tau) \end{Bmatrix} \tag{58, 59}$$

involving vector valued functions $\vec{X}(\vec{x},\tau)$, $\vec{C}(\vec{c},\tau)$. The next step shows that the right hand side of Eqs. (54)–(57) satisfies a Lipschitz condition. Referring to Eq. (39) and carrying out the Lipschitz test for the cylinder, i.e. in Eqs. (54)–(57), and considering this equation as a coupled vector, the Lipschitz inequality becomes

$$\left\| \begin{Bmatrix} X_1(x_2,\tau) - X_1(y_2,\tau) \\ X_2(x_1,x_2,c_1,\tau) - X_2(y_1,y_2,d_1,\tau) \\ C_1(c_2,\tau) - C_1(d_2,\tau) \\ C_2(c_1,c_2,x_1,x_2,\tau) - C_2(d_1,d_2,y_1,y_2,\tau) \end{Bmatrix} \right\| \leq \overline{L} \left\| \begin{Bmatrix} x_1 - y_1 \\ x_2 - y_2 \\ c_1 - d_1 \\ c_2 - d_2 \end{Bmatrix} \right\|. \tag{60}$$

Further, expanding the left-hand side of Eq. (60) and using Eqs. (54)–(57), we have

$$\left\| \begin{Bmatrix} x_2 - y_2 \\ (-\tilde{D}x_2 - x_1 + \tilde{G}c_1) - (-\tilde{D}y_2 - y_1 + \tilde{G}d_1) \\ c_2 - d_2 \\ (Dc_2 - Ec_2^3 - Fc_1 + Gx_1) - (Dd_2 - Ed_2^3 - Fd_1 + Gy_1) \end{Bmatrix} \right\|. \tag{61}$$

Next, on simplifying Eq. (61), we have

$$\left\| \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -D & \tilde{G} & 0 \\ 0 & 0 & 0 & 1 \\ G & 0 & -F & D - E(c_2^2 + c_2 d_2 + d_2^2) \end{bmatrix} \begin{Bmatrix} x_1 - y_1 \\ x_2 - y_2 \\ c_1 - d_1 \\ c_2 - d_2 \end{Bmatrix} \right\|. \tag{62}$$

Then, making use of the Schwarz Inequality for two vectors $\vec{A}$ and $\vec{B}$, which states

$$\|\vec{A} \cdot \vec{B}\| \leq \|\vec{A}\| \cdot \|\vec{B}\| \tag{63}$$

and comparing Eq. (62) with the left-hand side of Eq. (63), and referring to Eq. (39), we obtain the result that the one-norm of matrix A in Eq. (62) serves as a suitable Lipschitz constant. The one-norm is chosen since it is subordinate to the vector sum norm, which implies that the inequality stated in Eq. (63) will always be satisfied (Young and Gregory, 1973). It is to be noted here that the matrix one-norm of A is positive and larger than unity since by definition the one-norm of matrix A is

$$||A||_1 = \max_j \sum_{i=1}^{n} a_{ij}. \tag{64}$$

It must be mentioned here that there is no loss of generality in the above analysis which was carried out on the system of equations for the cylinder since the system of equations for the cylinder were the general form of the equation of motion and the Rayleigh equation.

5. Summary

For the model proposed by Skop and Balasubramanian (1997), it is not clear that for a stationary structure, where $\dot{y}$ is zero, the equation has a self-limiting solution, i.e.

$$\ddot{q} - \omega_s G(C_{L0}^2 - 4q^2)\dot{q} + \omega_s^2 q = \omega_s F \dot{y},$$

where q is proposed to take the form

$$q = C_{L0} \sin \omega_s t.$$

From observation, the $\dot{q}$ term does not cancel out and the statement that $C_{L0}^2 \ll 1$ does not hold for general cases, thus making the relation for a self-limiting solution invalid. Also, the governing Van der Pol oscillator equations are transcendental in nature; therefore, the current approach would be to generate numerical solutions rather than explore some quasi-closed form solutions as proposed by Facchinetti et al. (2004).

Basically from the literature reviewed, the form of the Rayleigh equation used by Skop and Balasubramanian (1997) and developed by Facchinetti et al. (2004) is of the classical form, and the ‘‘quasi’’ nonlinear term $\dot{q}; q$ indicates that the Lipschitz condition cannot be satisfied, and as a result the problem becomes ill-defined and a numerical solution may not be easily formulated. For this reason, the model proposed by Williams and Suaris (2006), is preferred since this model produces a well-defined problem and from actual research, in comparing with actual cylinders subjected to aerodynamic excitations, proves to converge to the observed results.

6. Conclusions

Convergence is taken to mean that the numerical or difference method applied to the solution of the differential equation approaches the exact solution to the differential equation as the step size approaches zero, assuming there is no round off error. In general, in applying difference methods to cases where the data is not well behaved in the classical sense, convergence is still achieved as discussed in Theorem 2.1 in Chartres and Stepleman (1971), and further elaborated as Theorem 1 in Feldstein and Goodman (1973).

In the classical situation, the convergence theorem of Young and Gregory (1973) shows that the truncation error or discretization error in Euler's method goes to zero as the step size goes to zero. In contrast, it can be shown (Young and Gregory, 1973) that, as the step size goes to zero, the round-off error increases. This means that the step size should not be reduced to approach zero indefinitely. Also, from the results of the analysis carried out in this paper, it is interesting to note that the finite difference approximations produce results which converge from below, i.e. lower bound solutions are obtained, or in other words the exact solution is greater than the approximate solution. This can be stated as

$$X_{\text{approx}} \leq X_{\text{exact}}. \quad (65)$$

This has both advantages and disadvantages from an engineering point of view.

The concept of stability discussed in this section is a joint property of the numerical method and the differential equations under investigation. For a convergent method, the fundamental solution approximates the exact solution as the time step goes to zero and the round-off error neglected. Now, when the fundamental solution is overpowered by parasitic solutions or solutions that feed on the errors in the numerical solution, i.e. truncation and round-off errors, then instability results.

For the analysis carried out, numerical solutions were obtained with varying step sizes, and these showed no significant differences; therefore the system was considered stable and there was no need to try alternative solution methods.

Acknowledgements

Rupert G. Williams was partially funded for this research under a fellowship from the Organization of American States. Richard H. Goodman was partly supported by a sabbatical leave from the University of Miami.

References

Arrowsmith, D.K., Place, C., 1982. Ordinary Differential Equations. Chapman and Hall.

Bearman, P.W., 1984. Vortex shedding from oscillating bluff bodies. Annual Review of Fluid Mechanics 16, 195–222.

Bishop, R.E.D., Hassan, A.Y., 1964. The lift and drag forces on a circular cylinder in a flowing fluid. Proceedings of the Royal Society of London A 277 (1368), 32–50.

Brika, D., Laneville, A., 1993. Vortex-induced vibrations of a long flexible circular cylinder. Journal of Fluid Mechanics 250, 481–508.

Chartres, B., Stepleman, R., 1971. Convergence of difference methods for initial and boundary value problems with discontinuous data. Mathematics of Computation 25 (116), 729–732.

Facchinetti, M.L., de Langre, E., Biolley, F., 2004. Coupling of structure and wake oscillators in vortex-induced vibrations. Journal of Fluids and Structures 19, 123–140.

Feldstein, A., Goodman, R., 1973. Numerical solution of ordinary and retarded differential equations with discontinuous derivatives. Numerische Mathematik 21, 1–13.

Goswami, I., Scanlan, R., Jones, N., 1993. Vortex-induced vibration of circular cylinders. I: experimental data. Journal of Engineering Mechanics 119 (11), 2270–2287.

Hartlen, R.T., Currie, I.G., 1970. Lift-oscillator model of vortex-induced vibration. Journal of the Engineering Mechanics Division ASCE 96 EM5, 577–591.

Kitagawa, T., Wakahara, T., Fujino, Y., Kimura, K., 1997. An experimental study on vortex-induced vibration of a circular cylinder tower at a high wind speed. Journal of Wind Engineering and Industrial Aerodynamics 69–71, 731–744.

Ohya, Y., Okajima, A., Hayashi, M., 1989. Wake interference and vortex shedding. In: Cheremisinoff, N. (Ed.), Encyclopedia of Fluid Mechanics, vol. 8. pp. 323–389.

Shih, W.C.L., Wang, C., Coles, D., Roshko, A., 1993. Experiments on flow past rough circular cylinders at large Reynolds numbers. Journal of Wind Engineering and Industrial Aerodynamics 49, 351–368.

Simiu, E., Scanlan, R., 1996. Wind Effects on Structures, Fundamentals and Applications to Design, third edition John Wiley New York.

Skop, R.A., Balasubramanian, S., 1997. A new twist on an old model for vortex-excited vibrations. Journal of Fluids and Structures 11, 395–412.

Williams, R.G., Suaris, W., 2006. An analytical approach to wake interference effects on circular cylindrical structures. Journal of Sound and Vibration 295, 266–281.

Young, D.M., Gregory, R., 1973. A Survey of Numerical Mathematics: 2 Volumes. Dover Publications, Inc., New York.